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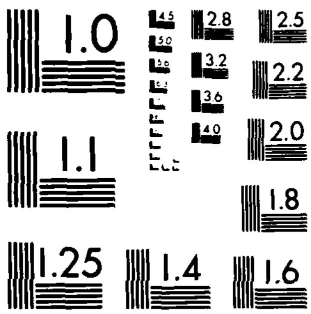

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DIRECTIONAL CONVEXITY AND FINITE OPTIMALITY CONDITIONS

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ABSTRACT

Summary. For $\Lambda \subseteq R^d$, we say that a set $A \subseteq R^d$ is Λ-convex if the segment $\overline{pq}$ is contained in A whenever $p, q \in A$ and $p - q \in \Lambda$. For the control system $\dot{x}(t) = G(x(t)) \cdot u(t)$, $x(o) = 0 \in R^d$, $u(t) \in \Omega \subseteq R^m$ for every $t \in [0,T]$, assuming that the reachable set $R(T)$ is Λ-convex, an extension of the Pontryagin Maximum Principle is proved. If $x(\bar{u},T)$ lies on the boundary of $R(T)$, conditions on the first order tangent cone at $x(\bar{u},T)$ can indeed be combined with restrictions placed on finite difference vectors of the form $y - x(\bar{u},T)$, with $y \in R(T)$, $y - x(\bar{u},T) \in \Lambda$. This result is complemented by conditions insuring the directional convexity of the reachable set. They rely on a uniqueness assumption for solutions of the Pontryagin equations and are proven by means of a generalized Mountain Pass Theorem.

AMS (MOS) Subject Classifications: 49E15, 58E25, 93C10

Key Words: Directional convexity, Critical point, Nonlinear control system, Necessary Conditions for optimality.

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SIGNIFICANCE AND EXPLANATION

Consider a control system of the form

$$\dot{x}(t) = \sum_{i=1}^{m} g_i(x(t))u_i(t)\ , \quad x(o) = 0 \in \mathbb{R}^d,$$

$$u(t) = (u_1(t), \ldots, u_m(t)) \in \Omega \subset \mathbb{R}^m \qquad \forall t \in [0,T].$$

Assume that an admissible control $\bar{u}(\cdot)$ steers the system to a point $x(\bar{u},t)$ on the boundary of the reachable set $R(t)$. Then, by studying the variations $x(u,t) - x(\bar{u},t)$ for controls u which are infinitesimally close to $\bar{u}$, one concludes that $\bar{u}$ must satisfy the Pontryagin Maximum Principle.

This paper establishes further necessary conditions for optimality, by considering finite difference vectors of the type $y - x(\bar{u},t)$, y being any point in $R(t)$, not necessarily close to $x(\bar{u},t)$. These conditions are significant whenever the reachable set is a priori known to be directionally convex. More precisely, we say that $R(t)$ is Λ-convex if a segment $\overline{pq}$ is entirely contained in $R(t)$ whenever its end points p,q lie in $R(t)$ and the direction of $p-q$ falls inside a preassigned cone of directions Λ. An efficient method for determining the Λ-convexity of a reachable set is developed.

The major application of the present results appears in [3], where they are used to determine the local time-optimal stablizing controls in a generic 3-dimensional problem.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

DIRECTIONAL CONVEXITY AND FINITE OPTIMALITY CONDITIONS

Alberto Bressan*

1. Introduction.

Consider a control system of the form

$$\dot{x}(t) = G(x(t))u(t), \quad x(o) = 0 \in \mathbb{R}^d, \tag{1.1}$$

where G is a $d\times m$ matrix with continuously differentiable entries and the control u lies in the admissible set

$$U = \{u = (u_1,\ldots,u_m) \in \mathcal{L}^2([0,T];\mathbb{R}^m);\ u(t) \in \Omega \ \text{ a.e.}\}, \tag{1.2}$$

Ω being a fixed compact convex subset of $\mathbb{R}^m$. Given a control $u \in U$, let $t \to x(u,t)$ be the corresponding solution of (1.1) and call $\mathcal{R}(t)$ the reachable set at time t. Several necessary conditions are known in order for a trajectory $x(\bar{u},\cdot)$ to reach a boundary point of $\mathcal{R}(T)$ at time T [2,7,8]. All of these conditions are obtained from a local analysis: to test the optimality of a control $\bar{u}(\cdot)$, a one-parameter family of control functions $u_\xi(\cdot)$ is constructed, which generates at time T a tangent vector

$$v = \lim_{\xi\to 0} [x(u_\xi,T) - x(\bar{u},T)]/\xi. \tag{1.3}$$

If, by choosing different families of controls $u_\xi(\cdot)$, one can generate tangent vectors $v_1, \ldots, v_n$ whose positive span is all of $\mathbb{R}^d$, then it can usually be shown that $x(\bar{u},T)$ lies in the interior of $\mathcal{R}(T)$. In the present paper we consider not only infinitesimal tangent vectors of the form (1.3), but also finite difference vectors:

$$w = y - x(\bar{u},T) \tag{1.4}$$

where y is any point in $\mathcal{R}(T)$. Assume that there exist first order tangent vectors $v_1,\ldots,v_n$ and finite difference vectors $w_1, \ldots, w_\nu$ such that the positive span of $\{v_1, \ldots, v_n, w_1, \ldots, w_\nu\}$ is all of $\mathbb{R}^d$. The a-priori assumption that $\mathcal{R}(T)$ is convex would then imply $x(\bar{u},T) \in \text{int}\ \mathcal{R}(T)$.

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Unfortunately, in a nonlinear setting such an assumption is far too restrictive for most of the relevant applications. A less stringent condition is the following kind of directional convexity:

Definition. Let Λ be a subset of $\mathbb{R}^d$. A set $A \subseteq \mathbb{R}^d$ is Λ-convex if $p, q \in A$ and $p - q \in \Lambda$ imply $\xi p + (1-\xi)q \in A$ for all $\xi \in [0,1]$.

Using the notion of Λ-convexity we obtain an extension of the Pontryagin Maximum Principle for the system (1.1).

Theorem 1. Assume that the reachable set $R(T)$ is Λ-convex for some $\Lambda \subseteq \mathbb{R}^d$. Let $\bar{u} \in U$ be a control such that $x(\bar{u},T)$ lies on the boundary of (T). Then there exists a nontrivial adjoint vector $\lambda(\cdot)$ such that

$$\dot{\lambda}(t) = -\lambda(t) \cdot G_x(x(\bar{u},t)) \cdot \bar{u}(t), \tag{1.5}$$

$$\langle \lambda(t), G(x(\bar{u},t))\bar{u}(t) \rangle = \max \{\langle\lambda(t),G(x(\bar{u},t))\cdot u\rangle \, ; \, u \in \Omega\}$$

$$\text{a.e. in } [0,T], \tag{1.6}$$

$$\langle \lambda(T), y - x(\bar{u},T) \rangle \leq 0 \tag{1.7}$$

for all $y \in R(T)$ such that $y - x(\bar{u},T)$ lies in the interior of Λ.

Here (1.5) and (1.6) are a restatement of the Maximum Principle, while the non-positivity of the inner product in (1.7) poses an additional requirement whenever the Λ-convexity of the reachable set is a-priori known. Of course, Theorem 1 would have little significance unless we provide some efficient way to determine the Λ-convexity of (T) for some set Λ. This is indeed the major concern of the present paper. Let the sets $R(t)$ be bounded for $0 \leq t \leq T$. Then for all nontrivial $\eta \in \mathbb{R}^d$ there exists at least one control $u_\eta(\cdot)$ for which the Pontryagin equations

$$\begin{aligned} &\dot{x}(t) = G(x(t))\, u(t) \\ &\dot{\lambda}(t) = -\lambda(t) \cdot G_x(x(t))\cdot u(t) \\ &x(o) = 0, \ \lambda(T) = \eta \\ &\langle \lambda(t), G(x(t))u(t)\rangle = \max \{\langle\lambda(t),G(x(t))u \rangle; \ u \in \Omega\} \end{aligned} \tag{1.8}$$

are a.e. satisfied. Indeed $R(T)$ is compact, hence the problem

$$\max_{u \in U} \langle \eta, x(u,T) \rangle \tag{1.9}$$

has at least one solution. The control u_η which attains the maximum clearly yields a solution to (1.8). We will infer the directional convexity of $\mathcal{R}(T)$ from the uniqueness of the solution of (1.8) for certain η, relying on a version of the Mountain Pass Theorem [1,4]. In the following we call $w^\perp$ the hyperplane orthogonal to the vector $w \in R^d$. Our main result is

Theorem 2. Assume that the reachable sets $\mathcal{R}(t)$, $0 < t < T$, for the control system (1.1) are uniformly bounded. Then $\mathcal{R}(T)$ is Λ-convex, Λ being the set of all $w \in R^d$ such that for every nontrivial $\eta \in w^\perp$

a) the equations (1.8) have a unique solution, say $(\bar{u}(\cdot), \bar{x}(\cdot), \bar{\lambda}(\cdot))$,

b) for a.e. $t \in [0,T]$, $\bar{u}(t)$ is the unique point in Ω where the map $u \to \langle \bar{\lambda}(t), G(\bar{x}(t))\cdot u \rangle$ attains its maximum.

Notice that, for the set Λ defined in Theorem 2, $w \in \Lambda$ always implies $\xi w \in \Lambda$ for every $\xi \in R$, $\xi \neq 0$. Λ *thus represents a family of* directions. This motivates the term "directional convexity" used throughout the paper.

The major application of the present results appears in [2], where the above theorems rule out the optimality of certain bang-bang controls for which Pontryagin's test is inconclusive. This leads to the solution of the generic local time-optimal stabilization problem for systems of the type $\dot{x} = X(x) + Y(x)u$, in dimension 3.

2. Preliminaries

Consider a mapping ϕ from a Hilbert space H into R^d. Its Frechet differential at a point $u \in H$ is denoted by $D\phi(u)$. We say that ϕ is C^1 if the map $u \to D\phi(u)$ from H into the space of continuous linear operators $L(H;R^d)$ from H into R^d is continuous. For the definition of the operator norm on $L(H,R^d)$ and for the basic properties of differential calculus in abstract spaces our general reference is Dieudonne

[6]. If $f : H \to R$ is Lipschitz continuous, the generalized directional derivative of f at $\bar{u}$ in the direction v is

$$f^{o}(\bar{u};v) = \overline{\lim_{\substack{u \to \bar{u} \\ \xi \downarrow 0}}} \xi^{-1}[f(u+\xi v) - f(u)],$$

and the generalized gradient of f at $\bar{u}$, denoted $\partial f(\bar{u})$, is the subdifferential of the convex function $v \to f^{o}(\bar{u},v)$ at the origin [5]. Thus $w \in \partial f(\bar{u})$ if, for all $v \in H$ $< w,v > \leq f^{o}(\bar{u},v)$. If U is a closed convex subset of H, we denote by $\Gamma(u)$ and $\Gamma^{\perp}(u)$ the tangent and the normal cone to U at a point $u \in U$. If $M > 0$ and $d_U(v)$ denotes the distance from a point $v \in H$ to U, the generalized gradient of the map $v \to M \cdot d_U(v)$ at $\bar{u} \in U$ is

$$\partial(M \cdot d_u)(\bar{u}) = \{w \in \Gamma^{\perp}(\bar{u});\ \|w\| \leq M\}. \tag{2.1}$$

We write $B(x,r)$ for the closed ball centered at x with radius r, int A and $\overline{co}\,A$ for the interior and the convex closure of the set A. Consider now the special case where $H = L^2([0,T];R^m)$, U is the convex set defined at (1.2) and $\phi : L^2 \to R^d$ is the map $u(\cdot) \to x(u,T)$ generated by (1.1). Then ϕ is continuously differentiable (see [2] for details) and $D\phi(\bar{u})$ is the linear map

$$u(\cdot) \to \int_0^T M(T,s)\, G(x(\bar{u},s))u(s)ds \tag{2.2}$$

where $s \to M(T,s)$ is the dxd matrix fundamental solution of

$$\dot{z}(t) = -z(t) \cdot G_x(x(\bar{u},t))\bar{u}(t)$$

with $M(T,T) = I$. Here G_x denotes the differential of the map $G : R^d \to R^{dxm}$ in (1.1). For $\eta \in R^d$, the differential of the map $u(\cdot) \to < \eta, \phi(u) >$ at $\bar{u}$ is the linear map

$$u(\cdot) \to \int_0^T \lambda(s)\, G(x(\bar{u},s))u(s)ds, \tag{2.3}$$

$\lambda(\cdot)$ being the unique solution to

$$\dot{\lambda}(t) = -\lambda(t)\, G_x(x(\bar{u},t))\bar{u}(t)\ , \quad \lambda(T) = \eta \tag{2.4}$$

In this case, if $\bar{u}(\cdot) \in U$, $\Gamma^{\perp}(\bar{u})$ is the set of L^2 functions $z(\cdot)$ such that

$$\int_0^T z(s)\,(u(s) - \bar{u}(s))ds \leq 0 \quad \forall u \in U. \tag{2.5}$$

3. Proof of Theorem 1.

Consider a control $\bar{u} \in U$, and suppose that the conclusion of Theorem 1 does not hold for $\bar{u}$. Then for every nontrivial vector $\eta \in R^d$ one of the following alternatives holds:

i) if $\lambda(\cdot)$ denotes the unique solution of (1.5) for which $\lambda(T) = \eta$, then

$\lambda(t)\, G(x(\bar{u},t))\bar{u}(t) < \max \{\lambda(t)G(x(\bar{u},t))u;\ u \subset \Omega\}$

for t in a subset $J \subseteq [0,T]$ of positive measure, or

ii) there exists some $y \in (T)$ with

$w = y - x(\bar{u},T) \in \operatorname{int} \Lambda$ and $\langle \eta, y - x(\bar{u},T) \rangle > 0$.

In the first case, let $u_\eta(\cdot)$ be a control in U such that

$\lambda(t)\cdot G(x(\bar{u},t))u_\eta(t) = \max \{\lambda(t)\cdot G(x(\bar{u},t))u;\ u \in \Omega\}$

for a.e. $t \in [0,T]$. For $\xi \in [0,1]$, the convex combination $\xi u_\eta(\cdot) + (1-\xi)\bar{u}(\cdot)$ is again an admissible control. It was shown in [2] that

$v = \lim\limits_{\xi \to 0} [x(\xi u_\eta + (1-\xi)\bar{u},T) - x(\bar{u},T)]/\xi$

is a first order tangent vector whose inner product with η is strictly positive. A standard argument now implies that R^d is the positive span of $d+1$ vectors : $v_1, \ldots, v_n, w_1, \ldots, w_\nu$ with the following property:

For $i = 1, \ldots, n$ there exists a control $u_i(\cdot) \in U$ such that

$$v_i = \lim_{\xi \to 0} [x(\xi u_i + (1-\xi)\bar{u},T) - x(\bar{u},T)]/\xi = D\,\phi(\bar{u}) \cdot (u_i - \bar{u})$$

with $D\,\phi(\bar{u})$ given at (2.2), and for $j = 1, \ldots, \nu$ there exists a point $y_j \in R(T)$ such that $w_j = y_j - x(\bar{u},T)$.

Moreover, w_i lies in the interior of Λ. By constructing suitable convex combinations of the vectors v_i, w_j we will show that the above assumptions imply $x(\bar{u},T) \in \operatorname{int} R(T)$. Define

$$\Delta^{n+\nu} = \{(c_1, \ldots, c_n, c_{n+1}, \ldots c_{n+\nu}) \in R^{n+\nu};\ c_i \geq 0,\ \sum_{i=1}^{n+\nu} c_i = 1\}.$$

For $c \in \Delta^{n+\nu}$, $\xi \in [0,1]$ set

$p_o(c,\xi) = x((1-\xi)\bar{u} + \xi \sum_{i=1}^{n} c_i(u_i-\bar{u}), T)$.

By induction on $j = 1, \ldots, \nu$ define

$$p_j(c,\xi) = p_{j-1}(c,\xi) + \xi c_{n+j}(y_j - p_{j-1}(c,\xi))$$

and set $p(c,\xi) = p_\nu(c,\xi)$. Let $\rho > 0$ be such that $B(w_j,\rho) \subsetneq \Lambda$ for all $j = 1, \ldots, \nu$. Let $M = \max\{|w_j|; j = 1, \ldots, \nu\}$. Choose $\bar{\xi} > 0$ so small that $\bar{\xi} < \rho\,[2\nu(M+\rho)]^{-1}$ and $|p_o(c,\xi) - x(\bar{u},T)| < \rho/2$ for all $(c,\xi) \in \Delta^{n+\nu} \times [0,\bar{\xi}]$. An easy inductive argument now yields

$|p_j(c,\xi) - x(\bar{u},T)| < \rho/2 + \frac{j\rho}{2\nu} < \rho$

for all c, ξ, j, hence $|y_j - p_{j-1} - w_j| < \rho$. If $p_{j-1}(c,\xi) \in R(T)$, then also $p_j(c,\xi) \in R(T)$, due to the Λ-convexity of the reachable set. By induction on j it follows that $p(c,\xi) \in R(T)$ for all $(c,\xi) \in \Delta^{n+\nu} \times [0,\bar{\xi}]$. The continuous Frechet differentiability of the map $\phi : u(\cdot) \to x(u,t)$ implies that

$$\begin{aligned}
p_o(c,\xi) - x(\bar{u},T) &= \int_0^\xi D\phi(\bar{u}+\zeta \sum_{i=1}^{n} c_i(u_i-\bar{u})) \cdot \sum_{i=1}^{n} c_i(u_i-\bar{u})d\zeta \\
&= \xi \cdot D\phi(\bar{u}) \cdot \sum_{i=1}^{n} c_i(u_i-\bar{u}) \\
&\quad + \int_0^\xi \left[D\phi(\bar{u}+\zeta \sum_{i=1}^{n} c_i(u_i-\bar{u})) - D\phi(\bar{u})\right] \cdot \sum_{i=1}^{n} c_i(u_i-\bar{u})d\zeta \\
&= \xi \sum_{i=1}^{n} c_i v_i + o(\xi) \qquad (3.1)
\end{aligned}$$

with $\lim_{\xi \to 0} o(\xi)/\xi = 0$ uniformly w.r.t. $c \in \Delta^{n+\nu}$.

Moreover

$$\lim_{\xi \to 0} [p_j(c,\xi) - p_{j-1}(c,\xi)]/\xi = c_{n+j} \lim_{\xi \to 0} (y_j - p_{j-1}(c,\xi)) = c_{n+j}\, w_j \qquad (3.2)$$

also holds uniformly w.r.t. $c \in \Delta^{n+\nu}$. Together, (3.1) and (3.2) yield

$$\lim_{\xi \to 0} [p(c,\xi) - x(\bar{u},T)]/\xi = \sum_{i=1}^{n} c_i v_i + \sum_{j=1}^{\nu} c_{n+j}\, w_j$$

uniformly w.r.t. $c \in \Delta^{n+\nu}$. The argument that completes the proof is now well established. Let δ be the distance of $0 \in \mathbb{R}^d$ from the boundary of $\overline{co}\,\{v_1,\ldots,v_n,w_1,\ldots,w_\nu\}$ and choose $\xi_o > 0$ so small that $\varepsilon_o < \bar{\xi}$ and

$$|p(c,\xi_o) - x(\bar{u},T) - \xi_o\,(\sum_{i=1}^{n} c_i v_i + \sum_{j=1}^{\nu} c_{n+j}\, w_j)| < \xi_o \cdot \delta/2 \tag{3.3}$$

for all $c \in \Delta^{n+\nu}$. Consider the injective map $\sigma : \Delta^{n+\nu} \to \mathbb{R}^d$ defined by

$$\sigma(c) = x(\bar{u},T) + \xi_o(\sum_{i=1}^{n} c_i v_i + \sum_{j=1}^{\nu} c_{n+j}\, w_j).$$

For $z \in B(x(\bar{u},T),\ \xi_o\delta)$ define $F(x) = p(\sigma^{-1}(z),\ \xi_o)$. By (3.3), $|F(z) - z| < \xi_o \cdot \delta/2$. For each $z_o \in B(x(\bar{u},T),\ \xi_o \cdot \delta/2)$, an application of Brouwer's theorem ([7] pg. 251) now implies the existence of some $z \in B(x(\bar{u},T),\ \xi_o\delta)$ for which $F(z) = z_o$. Hence $B(x(\bar{u},T),\ \xi_o\delta/2) \subseteq \mathcal{R}(T)$, Q.E.D.

4. An abstract result on directional convexity

The additional condition (1.5) stated in Theorem 1 is useful only if one can prove that the reachable set $\mathcal{R}(T)$ is Λ-convex for some (hopefully large) set $\Lambda \subseteq \mathbb{R}^d$. Recasting the problem in a more general setting, we now examine the directional convexity of the image $\psi(U)$ of a convex set U under an arbitrary differentiable map ψ.

Theorem 3. Let U be a closed, convex, bounded set in a Hilbert space H and let ψ be a C^1 map from a neighborhood of U into $\mathbb{R}^d$, with $\psi(U)$ compact and $\|D\psi\|$ uniformly bounded on U. Then $\psi(U)$ is Λ-convex, Λ being the set of vectors $w \in \mathbb{R}^d$ such that, for every nontrivial $\eta \in w^\perp$,

a') there exists a unique $\bar{u} \in U$ for which

$$\langle \eta,\ D\psi(\bar{u}) \cdot y \rangle \le 0 \qquad \text{for all} \quad y \in \Gamma(\bar{u}),$$

b') If $u_n \in U$, $z_n \in \Gamma^\perp(U_n)$, $n = 1, 2, \ldots$ and $\|\eta \cdot D\psi(u_n) - z_n\| \to 0$, then the sequence $(u_n)_{n\geq 1}$ has a convergent subsequence.

Notice that in condition b') both z_n and the map $v \to \langle \eta, D\psi(u_n) v \rangle$ are continuous linear functionals on H, hence the norm of their difference is well defined. To prove the theorem let $p' = \psi(u')$, $p'' = \psi(u'') \in R^d$, $u', u'' \in U$, $p''-p' = w \in \Lambda$. If the segment joining p' to p'' is not entirely contained in $\psi(U)$, the compactness of $\psi(U)$ implies

$$B(\bar{\xi}p' + (1-\bar{\xi})p'', \rho) \cap \psi(U) = \emptyset \tag{4.1}$$

for some $\bar{\xi} \in (0,1)$, $\rho > 0$. Let $\|D\psi(u)\| \le M_o$ for every u in a neighborhood U_o of U, and extend ψ from U_o to a function, still called ψ, defined and globally Lipschitz continuous on the whole space H, so that, say

$$|\psi(u_1) - \psi(u_2)| \le M \|u_1-u_2\| \qquad \forall\, u_1,u_2 \in H. \tag{4.2}$$

Let $d_\ell(p)$ be the distance of the point $p \in R^d$ from the line ℓ through p', p'' and denote by $d_U(v)$ the distance of $v \in H$ from the convex set U. Define the scalar functional f on H by

$$f(v) = d_\ell(\psi(v)) + 2Md_U(v). \tag{4.3}$$

Notice that f is globally Lipschitz continuous, with Lipschitz constant $3M$. By setting

$$m(v) = \inf \{\|y\|;\ y \in \partial f(v)\},$$

the elementary properties of generalized gradients [5] imply

$$m(v) \ge M > 0 \qquad \forall v \in U. \tag{4.4}$$

Let Σ be the set of all continuous paths $\gamma : [0,1] \to H$ with $\gamma(0) = p'$, $\gamma(1) = p''$.

Set

$$\bar{c} = \inf_{\gamma\in\Sigma} \sup_{\xi\in[0,1]} f(\gamma(\xi))$$

and observe that $c \ge \rho$. Indeed, if $\gamma \in \Sigma$, by (4.1) to (4.3)

$$\sup_{\xi\in[0,1]} f(\gamma(\xi)) \ge \sup_{\xi\in[0,1]} f(\pi_U(\gamma(\xi))) \ge \rho,$$

π_U being the orthogonal projection on U. Our next goal is to apply the deformation lemma, proved in [4] for Lipschitz continuous functionals, and conclude that $\bar{c}$ is a critical value for f. We first check that the Palais-Smale condition is satisfied.

Lemma 1 Any sequence $(u_n)_{n\geq 1}$ in H such that $f(u_n) \geq \rho/2$ and $m(u_n) \to 0$ possesses a convergent subsequence.

Proof. If $u_n \in U$, by (4.4) $m(u_n) \geq M$. We can thus assume that $u_n \in U$ for all $n \geq 1$. Let $\pi_\ell(x)$ be the orthogonal projection of $x \in R^d$ on the line ℓ through p', p''. For each $n \geq 1$, define the unit vector

$$\eta_n = [\ \pi_\ell(\psi(u_n)) - \psi(u_n)] \ / \ d_\ell(\psi(u_n)) \ . \tag{4.5}$$

The denominator in (4.5) is no less then $\rho/2$, hence η_n is well defined. Moreover $\eta_n \in (p''-p')^\perp = w^\perp$. By possibly taking a subsequence, we can assume that η_n converges to some unit vector $\eta \in w^\perp$. Relying on the assumption b'), the lemma will be proved by exhibiting a sequence $(z_n)_{n\geq 1}$ such that $\|\eta \cdot D\psi(u_n) - z_n\|$ converges to zero. The generalized gradient of f at u_n is

$$\partial f(u_n) = -\eta_n \cdot D\psi(u_n) + S_n,$$

with $S_n = \{y \in \Gamma^\perp(u_n);\ \|y\| \leq 2M\}$. Hence $m(u_n) \to 0$ implies that the distance between $\eta_n \cdot D\psi(u_n)$ and the set S_n tends to zero. Let z_n be the projection of $\eta_n \cdot D\psi(u_n)$ on S_n. Then $z_n \in \Gamma^\perp(u_n)$ and

$$\overline{\lim_{n\to\infty}} \|\eta \cdot D\psi(u_n) - z_n\| \leq \overline{\lim_{n\to\infty}} (\|\eta \cdot D\psi(u_n) - \eta_n \cdot D\psi(u_n)\| + \|\eta_n \cdot D\psi(u_n) - z_n\|) = 0.$$

By b') some subsequence of $(u_n)_{n\geq 1}$ converges.

To complete the proof of Theorem 3, for $c \in R$ define the sets

$$A_c = \{\ u \in H;\ f(u) \leq c\ \}$$

$$K_c = \{\ u \in H;\ f(u) = c,\ 0 \in \partial f(u)\ \}.$$

If $K_{\bar{c}}$ is empty, a generalized deformation lemma (Theorem 3.1 in [4]) yields the existence of a homeomorphism $h : H \to H$ such that

$$h(u) = u \quad \text{for } u \in A_{\bar{c}+\rho/2} \quad A_{\bar{c}-\rho/2}\ , \tag{4.6}$$

$$h(A_{\bar{c}+\varepsilon}) \subset A_{\bar{c}-\varepsilon} \quad \text{for some } \varepsilon > 0. \tag{4.7}$$

If $\gamma \in \Sigma$ is a path for which

$$\max_{\xi\in[0,1]} f(\gamma(\xi)) \leq \bar{c} + \varepsilon,$$

then the path $\gamma'(\cdot) = h(\gamma(\cdot))$ lies in Σ and

$$\max_{\xi\in[0,1]} f(\gamma'(\xi)) \le \bar{c} - \varepsilon,$$

contrary to the definition of $\bar{c}$. This proves the existence of some $\bar{u} \in H$ for which $0 \in \partial f(\bar{u})$ and $f(\bar{u}) = \bar{c} > \rho$. By (4.4), $\bar{u} \in U$. Setting

$\bar{\eta} = [\pi_{\ell}(\phi(\bar{u})) - \phi(\bar{u})]/d_{\ell}(\phi(\bar{u}))$ one has

$0 \in \partial f(\bar{u}) \subset \Gamma^{\perp}(\bar{u}) - \bar{\eta}\cdot D\phi(\bar{u})$.

This means $\bar{\eta}\cdot D\phi(\bar{u}) \in \Gamma^{\perp}(\bar{u})$, hence $< D\phi(\bar{u})\cdot y, \bar{\eta} > \le 0$ *for all* $y \in \Gamma(\bar{u})$. *On the other* hand, the compactness of the reachable set $\phi(U)$ implies the existence of some $\hat{u} \in U$ for which

$< \phi(\hat{u}), \bar{\eta} > = \max \{ < \phi(u), \bar{\eta} > ; u \in U\}$.

Since ϕ is continuously differentiable on a neighborhood of U, $\hat{u}$ satisfies the necessary conditions for optimality, i.e. $< D\phi(\hat{u})\cdot y, \bar{\eta} > \le 0$ for all $y \in \Gamma(\hat{u})$.

However, $\hat{u} \ne \bar{u}$ because

$< D\phi(\hat{u}), \bar{\eta} > \ge < p', \bar{\eta} > \ge < \phi(\bar{u}), \bar{\eta} > + \rho$.

This contradicts the uniqueness assumption a') and proves the theorem.

5. Proof of Theorem 2

Since G is C^1 and the sets $R(t)$ are uniformly bounded for $0 \le t \le T$, it is known that the map $\phi : u(\cdot) \to x(u,T)$ is continuously Frechet differentiable. By (2.2), $\|D\phi(u)\|$ is uniformly bounded as u ranges on a suitable small neighborhood of U. Moreover, $\phi(U) = R(T)$ is compact. Let $w \in \Lambda$ and $\eta \in w^{\perp}$, $\eta \ne 0$. It will be shown that ϕ satisfies the assumptions a'), b') stated for ϕ in Theorem 3. Let $\bar{u}(\cdot) \in U$ be an admissible control for which

$$< D\phi(\bar{u})\cdot y, \eta > \le 0 \qquad \forall y \in \Gamma(\bar{u}) \tag{5.1}$$

Let $u(\cdot)$ be any control in U and take $y(\cdot) = u(\cdot) - \bar{u}(\cdot) \in \Gamma(\bar{u})$. Using (2.3), (5.1) yields

$$\int_0^T \lambda(s)\, G(x(\bar{u},s))\, (u(s) - \bar{u}(s))ds \le 0 \tag{5.2}$$

with $\lambda(\cdot)$ defined at (2.4). Since $u(\cdot)$ was an arbitrary control in U, $\bar{u}(\cdot)$ yields a solution of the Pontryagin equations (1.8). The u⸱ jueness assumption a) thus implies a').

Now consider a sequence of controls $u_n(\cdot) \in U$ and linear functionals on $L^2([0,T];R^m)$, say $z_n(\cdot)$, with $z_n \in \Gamma^{\perp}(u_n)$, $n = 1, 2, \ldots,$ such that

$$\lim_{n\to\infty} \|\eta\cdot D\phi(u_n) - z_n\| = 0. \tag{5.3}$$

Setting $\lambda_n(\cdot)$ to be the unique solution of

$$\dot{\lambda}_n(t) = -\lambda_n(t)\ G_x(x(u_n,t))u_n(t),\quad \lambda_n(T) = \eta, \tag{5.4}$$

condition (5.3) can be written as

$$\lim_{n\to\infty} \Big[\int_0^T \lambda_n(t)G(x)(u_n,t))\cdot v_n(t)dt \\ - \int_0^T z_n(t)\ v_n(t)\ dt\Big] = 0 \tag{5.5}$$

for every bounded sequence $(v_n)_{n\geq 1}$ in $L^2([0,T];R^m)$.

Since $u_n(t) \in \Omega$ for all n and a.e. $t \in [0,T]$, the trajectories $x_n(\cdot) = x(u_n,\cdot)$ are uniformly Lipschitz continuous, and the same is true for the adjoint vectors $\lambda_n(\cdot)$. By possibly taking a subsequence and relabeling, we can assume that $x_n(\cdot) \to \bar{x}(\cdot)$ and $\lambda_n(\cdot) \to \bar{\lambda}(\cdot)$ in the norm topology of $C^0([0,T];R^d)$, while $u_n(\cdot) \to \bar{u}(\cdot)$ weakly, for some $\bar{x}(\cdot)$, $\bar{\lambda}(\cdot)$, $\bar{u}(\cdot)$. For every $t \in [0,T]$

$$x_n(u_n,t) = \int_0^t G(x_n(s))u_n(s)ds,$$

$$\lambda_n(t) = \eta + \int_t^T \lambda_n(s)\ G_x(x_n(s))\ u_n(s)ds.$$

Letting $n \to \infty$ in the above equalities, we obtain

$$\bar{x}(t) = \int_0^t G(\bar{x}(s))\bar{u}(s)ds, \tag{5.6}$$

$$\bar{\lambda}(t) = \eta + \int_t^T \bar{\lambda}(s)\ G_x(\bar{x}(s))\bar{u}(s)ds. \tag{5.7}$$

Therefore $\bar{x}(\cdot)$ is actually the trajectory of (1.1) corresponding to the control $\bar{u}(\cdot)$ and $\bar{\lambda}(\cdot)$ solves the correct adjoint equation in (1.8). The maximality condition in (1.8) also holds. Indeed, let $u(\cdot) \in U$. Since U is bounded, such is the sequence $v_n(\cdot) = u(\cdot) - u_n(\cdot)$. Thus (5.5) implies

$$0 = \lim_{n\to\infty} \Big[\int_0^T \lambda_n(t)\, G(x(u_n,t))\cdot(u(t) - u_n(t))dt \quad - \int_0^T z_n(t)\, (u(t) - u_n(t))dt \Big]$$

$$\geq \int_0^T \bar{\lambda}(t)\, G(\bar{x}(t))\, (u(t)-\bar{u}(t))dt \tag{5.8}$$

because, by (2.5), $\int_0^T z_n(t)\,(u(t) - u_n(t))dt \leq 0$ for all $n \geq 1$. Since $u(\cdot) \in U$ was arbitrary, (5.8), together with (5.6) and (5.7), shows that $\bar{u}(\cdot)$, $\bar{x}(\cdot)$, $\bar{\lambda}(\cdot)$ afford a solution to (1.8). Therefore the strong uniqueness condition b) can now be invoked. Intuitively, b) states that $\bar{u}(\cdot)$ is an exposed point of U, hence the convergence $u_n(\cdot) \to u(\cdot)$ takes actually place in the norm topology of $L^2([0,T];\ R^m)$ as well. A precise argument runs as follows. Suppose that $u_n(\cdot)$ did not converge to $\bar{u}$ in the L^2-norm. Then, because of the boundedness of Ω, there exist $\varepsilon > 0$, a subsequence u_ν and a set $J \subseteq [0,T]$ with positive Lebesque measure such that $|u_\nu(t) - \bar{u}(t)| \geq \varepsilon$ for $t \in J$ and all $\nu \geq 1$. For $t \in J$, set

$$\delta(t) = \langle \bar{\lambda}(t), G(\bar{x}(t))\bar{u}(t) \rangle - \max \{ \langle \bar{\lambda}(t), G(\bar{x}(t))u \rangle ;\ u \in \Omega,\ |u - \bar{u}(t)| \geq \varepsilon \} .$$

$\delta(\cdot)$ is measurable and strictly positive a.e. on J, hence

$$\bar{\delta} = \int_J \delta(t)dt > 0.$$

For each $\nu \geq 1$, define $v_\nu(t) = \bar{u}(t) - u_\nu(t)$ if $t \in J$, $v_\nu(t) = 0$ if $t \notin J$. Clearly $v_\nu(\cdot) \in \Gamma(u_\nu)$. Recalling that in (5.3) $z_n \in \Gamma^\perp(u_n)$, one has

$$\int_0^T \lambda_\nu(s)\, G\,(x_\nu(s))v_\nu(s)ds - \int_0^T z_\nu(s)v_\nu(s)ds$$

$$\geq \int_J \lambda_\nu(s)\, G(x_\nu(s))v_\nu(s)ds$$

$$\geq \int_J \bar{\lambda}(s)G(\bar{x}(s))\, v_\nu(s)ds - \int_J |\lambda_\nu(s)-\bar{\lambda}(s)|\cdot|G(\bar{x}(s))|\cdot|v_\nu(s)|ds$$

$$- \int_J |\lambda_\nu(s)|\cdot|G(x_\nu(s)) - G(\bar{x}(s))|\cdot|v_\nu(s)|ds \tag{5.9}$$

Since the sequence v_ν is bounded, by (5.5) the left-hand side of (5.9) tends to zero as $\nu \to \infty$. However, the first integral on the right-hand side of (5.9) is $\geq \bar{\delta}$ for all ν, while the other two integrals tend to zero. This yields a contradiction and establishes the strong convergence of the sequence u_n to $\bar{u}$. Theorem 3 can thus be applied, completing the proof.

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Summary. For $\Lambda \subseteq \mathbb{R}^d$, we say that a set $A \subseteq \mathbb{R}^d$ is Λ-convex if the segment $\overline{pq}$ is contained in A whenever $p, q \in A$ and $p - q \in \Lambda$. For the control system $\dot{x}(t) = G(x(t))\cdot u(t)$, $x(0) = 0 \in \mathbb{R}^d$, $u(t) \in \Omega \subseteq \mathbb{R}^m$ for every $t \in [0,T]$, assuming that the reachable set $R(T)$ is Λ-convex, an extension of the Pontryagin Maximum Principle is proved. If $x(u,T)$ lies on the boundary of $R(T)$, conditions on the first order tangent cone at $x(u,T)$ can

ABSTRACT (Continued)

indeed be combined with restrictions placed on finite difference vectors of the form $y - x(\bar{u},T)$, with $y \in R(T)$, $y - x(\bar{u},T) \in \Lambda$. This result is complemented by conditions insuring the directional convexity of the reachable set. They rely on a uniqueness assumption for solutions of the Pontryagin equations and are proven by means of a generalized Mountain Pass Theorem.